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ASYMPTOTIC DETERMINATION OF THE FORMATION PROCESS OF NONLINEAR DISTORTION OF ONE-DIMENSIONAL PULSES IN A LAYERED MEDIUM

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Nonlinear effects in the propagation, reflection, and refraction of one-dimensional pulses in a medium consisting of two layers lying on a half-space are considered and analyzed. Properties of layers and of the half-space are different, and stresses are defined by an expansion in powers of strains. The initial pulse of finite duration is specified in the form of boundary condition at the surface of the external layer either for the deformation or for the dislocation rate, and the problem of wave pattern when the initial pulse amplitude tends to zero, i.e. in the case of small nonlinear effects, is solved.

Problem is solved by the method of successive integration of nonhomogeneous linear wave equations, in which the solution of the linear problem is taken as the first approximation and the subsequent approximations are derived by approximating the nonlinear terms with the use of the preceding approximation.

The derived first approximation formulas make possible to solve the inverse problem of acoustic determination of the properties of a medium by the parameters of reflected pulses. It follows from these that the measurement of parameters, which define the reflected pulses reaching the first layer within the accuracy of basic components of the nonlinear distortion, widens the information on the properties of layers in comparison with the disregard of nonlinear effects (although the number of constants that determine the properties of medium is increased).

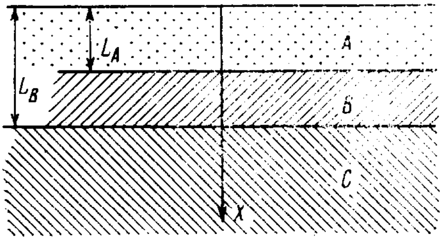

Fig. 1

1. Statement of the problem. Let us consider one-dimensional wave processes that depend on time t and the Lagrangian coordinate X with the dot and the prime denoting derivatives with respect to t and X, respectively. Let the finite intervals $0 \leqslant X \leqslant L_A$, $L_A \leqslant X \leqslant L_B$ and the semiinfinite interval $X \geqslant L_B$ be filled by different media A, B and C, respectively (see Fig. 1). Longitudinal dislocations in these intervals are denoted by $U_A(X, t)$, $U_B(X, t)$ and $U_C(X, t)$, longitudinal stresses by $\sigma_{11A}(X, t)$, $\sigma_{11B}(X, t)$ and $\sigma_{11C}(X, t)$, and densities in the initial state by ρ_A, ρ_B and ρ_C, respectively. Wave processes in the A, B and C media are, respectively, defined by the following equations:

$$[\sigma_{11j}(X, t)]' = \rho_j U_j^{\cdot\cdot}(X, t), \quad j = A, B, C \tag{1.1}$$

$$\sigma_{11j}(X, t) = Q_j(U_j'), \quad j = A, B, C \tag{1.2}$$

$$Q_j(U_j') = P_0 + \beta_j [U_j' + {}^1/_2\, k_j (U_j')^2 + {}^1/_3\, l_j (U_j')^3 + \ldots]$$

where P_0, ρ_j, β_j, k_j and l_j are constant coefficients.

We introduce the definitions

$$q_j(U_j') = \frac{1}{\beta_j}\frac{dQ_j(U_j')}{dU_j'}, \quad c_j = \left[\frac{\beta_j}{\rho_j}\right]^{1/2}, \quad j = A, B, C \tag{1.3}$$

From (1.1) with the use of (1.2) and (1.3) we obtain equations

$$c_j^{-2} U_j^{\cdot\cdot}(X, t) - q_j(U_j') U_j''(X, t) = 0, \quad j = A, B, C \tag{1.4}$$

$$q_j(U_j') = 1 + k_j U_j' + l_j (U_j')^2 + \ldots \tag{1.5}$$

and stipulate the following conditions.

1) Initial zero conditions

$$U_j(X, 0) = 0, \quad U_j^{\cdot}(X, 0) = 0, \quad j = A, B, C \tag{1.6}$$

2) One of the following two boundary conditions are specified along boundary $X = 0$:

$$U_A'(0, t) = \varepsilon\psi(t)\,[H(t) - H(t - t_0)] \quad \text{(problem 1)} \tag{1.7}$$

$$U_A^{\cdot}(0, t) = -\varepsilon\psi(t)\,[H(t) - H(t - t_0)]\, c_A \quad \text{(problem 2)} \tag{1.8}$$

where $H(t)$ is the Heaviside function, and t_0 and ε are constants that satisfy conditions

$$0 \leqslant t_0 < c_A^{-1} L_A, \quad c_B^{-1}(L_B - L_A); \quad |\varepsilon| \ll 1$$

and $\psi(t)$ is an arbitrary continuous function which satisfies conditions

$$\psi(0) = \psi(t_0) = 0, \quad \psi^{\cdot}(0) = \psi^{\cdot}(t_0) = 0$$
$$\max|\psi(t)| = 1, \quad 0 < t < t_0$$

and has in the interval $0 \leqslant t \leqslant t_0$ continuous derivatives of all orders required in the subsequent analysis.

3) Displacements and longitudinal stresses must be compatible at the interfaces of $X = L_A$ and $X = L_B$ of adjoining media, which with allowance for (1.2) yields the contact conditions

$$\begin{gathered} U_A(L_A, t) = U_B(L_A, t) \\ \beta_A \{U_A'(L_A, t) + {}^1\!/_2\, k_A [U_A'(L_A, t)]^2 + \ldots\} = \\ \beta_B \{U_B'(L_A, t) + {}^1\!/_2\, k_B [U_B'(L_A, t)]^2 + \ldots\} \end{gathered} \tag{1.9}$$

$$\begin{gathered} U_B(L_B, t) = U_C(L_B, t) \\ \beta_B \{U_B'(L_B, t) + {}^1\!/_2\, k_B [U_B'(L_B, t)]^2 + \ldots\} = \\ \beta_C \{U_C'(L_B, t) + {}^1\!/_2\, k_C [U_C'(L_B, t)]^2 + \ldots\} \end{gathered} \tag{1.10}$$

4) For problems 1 and 2 in which the (wave) processes are defined by (1.7) and (1.8), respectively, it is necessary to derive a solution that is asymptotic when $\varepsilon \to 0$ and determines the small deviation of the nonlinear solution from the linear one which is obtained by expanding functions $Q_j(U_j')$ ($j = A$, B and C) to within quadratic terms.

The sought solutions of Eqs. (1.4) which must satisfy the above conditions, represent the totality of pulses (see Fig. 2). The $U_{A(1)}$ is generated by the process at the edge $X = 0$, pulses $U_{A(2)}$ and $U_{B(1)}$ arise as the result of reflection-refraction $U_{A(1)}$ at the interface $X = L_A$ of media A and B, pulses $U_{B(2)}$ and $U_{C(1)}$ are the result of reflection-refraction $U_{B(1)}$ at the interface $X = L_B$ of media B and C and so on. We restrict the analysis to the pulses shown in Fig. 2.

We derive the solutions of problems 1 and 2 by the method of successive integration of linear nonhomogeneous wave equations [1 — 3]. The essence of that method consists of constructing zero approximations $U_{A(1)0}$, $U_{A(2)0}$, $U_{B(1)0}, \ldots$ of the considered pulses as solutions of the related linear problem. The subsequent approximations $U_{A(1)j}$, $U_{A(2)j}$, $U_{B(1)j}, \ldots$ ($j = 1, 2, 3, \ldots$) of the same pulses are derived by approximating the nonlinear terms in Eqs. (1.4), and also in the second contact conditions (1.9) and (1.10), using previous approximations. The calculations for the first approximation are presented

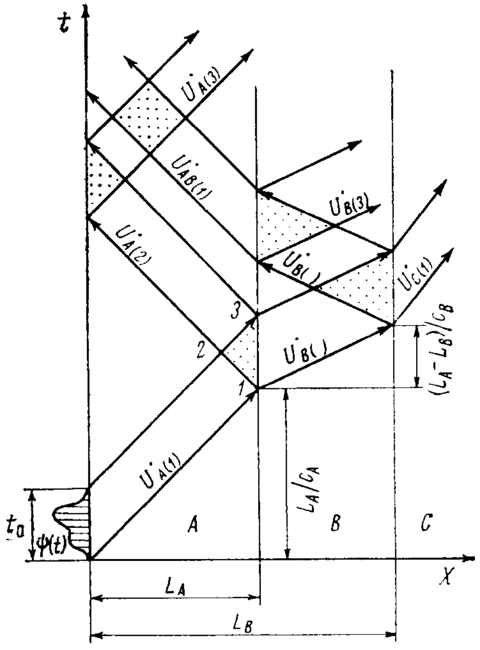

Fig. 2

below. To derive the latter it is necessary to determine the zero approximation, which is the same for both problems. Elementary reasoning will show that it can be represented in the form

$$U_{j(i)0}(X,t) = (-1)^i \varepsilon_{ij}[H(t_{ij}) - H(t_{ij}-t_0)] \int_0^{t_{ij}} \psi(z)\,dz + \qquad (1.11)$$

$$(-1)^i \varepsilon_{ij} H(t_{ij}-t_0) \int_0^{t_0} \psi(z)\,dz; \quad j = A, B, C; \quad i = 1, 2, 3$$

$$U_{AB(1)0}(X,t) = \varepsilon_{1AB}[H(t_{1AB}) - H(t_{1AB}-t_0)] \int_0^{t_{1AB}} \psi(z)\,dz + \qquad (1.12)$$

$$\varepsilon_{1AB} H(t_{1AB}-t_0) \int_0^{t_0} \psi(z)\,dz$$

In these formulas which relate to pulses shown in Fig. 2, the following notation is used:

$$\varepsilon_{1A} = \varepsilon c_A, \quad \varepsilon_{2A} = \varepsilon c_A J_A, \quad \varepsilon_{1B} = \varepsilon c_A(1 - J_A) \qquad (1.13)$$

$$\varepsilon_{2B} = \varepsilon c_A(1 - J_A) J_B, \quad \varepsilon_{3B} = -\varepsilon c_A(1 - J_A) J_A J_B$$

$$\varepsilon_{1C} = \varepsilon c_A(1 - J_A)(1 - J_B), \quad \varepsilon_{1AB} = \varepsilon C_A(1 - J_A^2) J_B$$

$$J_A = \frac{\alpha_A - 1}{\alpha_A + 1}, \quad J_B = \frac{\alpha_B - 1}{\alpha_B + 1} \qquad (1.14)$$

$$\alpha_A = \left[\frac{\beta_B \rho_B}{\beta_A \rho_A}\right]^{1/2}, \quad \alpha_B = \left[\frac{\beta_C \rho_C}{\beta_B \rho_B}\right]^{1/2}$$

$$t_{1A} = t - c_A^{-1} X, \quad t_{2A} = t - 2c_A^{-1} L_A + c_A^{-1} X \qquad (1.15)$$

$$t_{1B} = t - c_A^{-1} L_A - c_B^{-1}(X - L_A)$$

$$t_{2B} = t - c_A^{-1} L_A - 2c_B^{-1}(L_B - L_A) + c_B^{-1}(X - L_A)$$

$$t_{3B} = t - c_A^{-1} L_A - 2c_B^{-1}(L_B - L_A) - c_B^{-1}(X - L_A)$$

$$t_{1C} = t - c_A^{-1} L_A - c_B^{-1}(L_B - L_A) - c_C^{-1}(X - L_B)$$

$$t_{1AB} = t - c_A^{-1} L_A - 2c_B^{-1}(L_B - L_A) - c_A^{-1}(L_A - X)$$

In the limit case of absence of medium B, when ρ_B and β_B vanish, $\alpha_A = 0$ and, consequently, $J_A = -1$, while in the limit case of the absolutely rigid body, we have $\alpha_A \to \infty$ and $J_A = 1$. The limit values of J_B can be similarly elucidated. The nonlinear effects in the reflections of a pulse from a free and rigid boundary were considered in [2 — 7].

2. Asymptotic approximation of pulse $U_{A(1)}$. In [8] exact formulas and two forms of asymptotic representation for $\varepsilon \to 0$ are derived for calculating pulse $U_{A(1)}(X, t)$ and its first and second derivatives in problems 1 and 2 up to the instant of time $t = c_A^{-1} L_A$ at which begins the reflection of that pulse from the interface $X = L_A$. It was shown in [3, 8] that the asymptotic expansion of the exact solution along the characteristics of the linear wave equation are within the first and second approximations the same as those obtained earlier by the author in [2, 7] by the method of successive integration of nonhomogeneous linear wave equations. Hence only the first approximation formulas

$$U_{A(1)1}(X, t) = -\varepsilon_{1A}[H(t_{1A}) - H(t_{1A} - t_0)]\Big\{\int_0^{t_{1A}} \psi(z)\,dz + \frac{1}{8}\varepsilon_{1A} \times \qquad (2.1)$$
$$(1 + T_1) k_A c_A^{-1} \int_0^{t_{1A}} \psi^2(z)\,dz + \frac{1}{4}\varepsilon_{1A} k_A c_A^{-2} X \psi^2(t_{1A}) + \varepsilon^2(0)\Big\} -$$
$$\varepsilon_{1A} H(t_{1A} - t_0)\Big\{\int_0^{t_0} \psi(z)\,dz] + \frac{1}{8}\varepsilon_{1A}(1 + T_1) k_A c_A^{-1} \int_0^{t_0} \psi^2(z)\,dz + \varepsilon^2(0)\Big\}$$

are reproduced here. In these formulas and subsequently $T_1 = 1$ and $T_1 = -1$, respectively, for problems 1 and 2.

3. Asymptotic approximations of pulses $U_{A(2)}$ and $U_{B(1)}$. In the region of interaction between the incident pulse $U_{A(1)}$ and the reflected pulse $U_{A(2)}$ (see triangle 1—2—3 in Fig. 2) the sum $U_{A(1)} + U_{A(2)}$ must satisfy Eq. (1.4) with $j = A$ and pulse $U_{A(1)}$ has already been determined by the solution of that equation. Hence for the derivation of $U_{A(2)}$ we have the equation

$$\begin{aligned} c_A^{-2}\ddot{U}_{A(2)}(X, t) - U''_{A(2)}(X, t) = {} & [k_A U'_{A(2)} + l_A (U'_{A(2)})^2 + \\ & \ldots] U''_{A(2)} + [k_A U'_{A(2)} + 2 l_A U'_{A(1)} U'_{A(2)} + l_A (U'_{A(2)})^2 + \\ & \ldots] U''_{A(1)} + [k_A U'_{A(1)} + l_A (U'_{A(1)})^2 + 2 l_A U'_{A(1)} U'_{A(2)} + \\ & \ldots] U''_{A(2)} \end{aligned} \qquad (3.1)$$

The method of successive integration of nonhomogeneous linear wave equations reduces in the case of Eq. (3.1) to the successive integration of equations

$$c_A^{-2}\ddot{U}_{A(2)r}(X, t) - U''_{A(2)r}(X, t) = G_{A(2)r}(X, t), \quad = \; = 1, 2, 3, \ldots \qquad (3.2)$$

where

$$\begin{aligned} G_{A(2)r}(X, t) = {} & [k_A U'_{A(2)r-1} + l_A (U'_{A(2)r-1})^2 + \ldots] U''_{A(2)r-1} + \\ & [k_A U'_{A(2)r-1} + 2 l_A U'_{A(1)r-1} U'_{A(2)r-1} + l_A (U'_{A(2)r-1})^2 + \ldots] \times \\ & U''_{A(1)r-1} + [k_A U'_{A(1)r-1} + l_A (U'_{A(1)r-1})^2 + \\ & 2 l_A U'_{A(1)r-1} U'_{A(2)r-1} + \ldots] U''_{A(2)r-1} \end{aligned}$$

For the successive approximation of pulse $U_{B(1)}$ we obtain from Eq.(1.4) with $j = B$ the following nonhomogeneous linear wave equations:

$$c_B^{-2}\ddot{U}_{B(1)r}(X, t) - U''_{B(1)r}(X, t) = G_{B(1)r}(X, t), \quad r = 1, 2, 3, \ldots \qquad (3.3)$$

where

$$G_{B(1)r}(X, t) = [k_B U'_{B(1)r-1} + l_B (U'_{B(1)r-1})^2 + \ldots] U''_{B(1)r-1}$$

The integration of Eqs. (3.2) and (3.3) in each approximation $r = 1, 2, 3, \ldots$ must be carried out with the following conditions taken into account: pulse $U_{A(2)}$ is to propagate in the direction of decreasing X and pulse $U_{B(1)}$ in that of increasing X, the initial conditions (1.6) and the contact conditions (1.9) must be satisfied.

It is advisable to carry out calculations in two stages. In the first stage the integration of Eqs. (3.2) and (3.3) is carried out with the contact conditions (1.9) replaced, respectively, by conditions

$$\dot{U}_{A(2)r}(L_A, t) = \varepsilon_{2A}\psi_{2A}(t_{2A})\,[H(t_{2A}) - H(t_{2A} - t_0)] \qquad (3.4)$$

$$U_{B(1)r}(L_A,\ t) = -\varepsilon_{1B}\psi_{1B}(t_{1B})\ [H(t_{1B}) - H(t_{1B} - t_0)] \qquad (3.5)$$

where $\psi_{2A}(t_{2A})$ and $\psi_{1B}(t_{1B})$ are, so far, some unspecified functions.

At the second stage functions $\psi_{2A}(t_{2A})$ and $\psi_{1B}(t_{1B})$ are determined so that conditions (1.9) are satisfied.

This device makes it possible to consider separately in the first stage the problem of integrating Eqs. (3.2) and (3.3) separately. These problems were solved in [3] with the use of Laplace transformation. For brevity we omit intermediate computations and present their solutions in the first approximation as follows:

$$\begin{aligned}
U_{A(2)1}(X,t) = {} & \varepsilon_{2A}[H(t_{2A}) - H(t_{2A} - t_0)]\Big\{\int_0^{t_{2A}} \psi_{2A}(z)\,dz + \qquad (3.6)\\
& \frac{1}{4}\varepsilon_{2A}k_A c_A^{-2}(L_A - X)\psi^2(t_{2A}) - \frac{1}{4}\varepsilon_{1A}k_A c_A^{-1}H(t_{1A}) \times \\
& \Big[\psi(t_{1A})\int_0^{t_{2A}} \psi(z)\,dz - \psi(t_{2A})\int_0^{t_{1A}} \psi(z)\,dz\Big] + \frac{1}{4}\varepsilon_{1A}k_A c_A^{-1}H(t_{1A} - t_0) \times \\
& \Big[\psi(t_{1A})\int_0^{t_{2A}} \psi(z)\,dz - \psi(t_{2A})\int_{t_0}^{t_{1A}} \psi(z)\,dz\Big] + \varepsilon^2(0)\Big\} + \\
& \varepsilon_{2A}H(t_{2A} - t_0)\Big\{\int_0^{t_0} \psi_{2A}(z)\,dz + \varepsilon^2(0)\Big\}
\end{aligned}$$

$$\begin{aligned}
U_{B(1)1}(X,t) = {} & -\varepsilon_{1B}[H(t_{1B}) - H(t_{1B} - t_0)]\Big\{\int_0^{t_{1B}} \psi_{1B}(z)\,dz + \qquad (3.7)\\
& \frac{1}{4}\varepsilon_{1B}k_B c_B^{-2}(X - L_A)\psi_{1B}^2(t_{1B}) + \varepsilon^2(0)\Big\} - \varepsilon_{1B}H(t_{1B} - t_0) \times \\
& \Big\{\int_0^{t_0} \psi_{1B}(z)\,dz + \varepsilon^2(0)\Big\}
\end{aligned}$$

We pass to the execution of the second stage. Noting that for $X = L_A$ we have

$$U_A(L_A,\ t) = U_{A(1)}(L_A, t) + U_{A(2)}(L_A,\ t)$$
$$U_B(L_A,\ t) = U_{B(1)}(L_A,\ t)$$

and using formulas (2.1), (3.6) and (3.7), we can obtain from contact condition (1.9) the following expressions for functions ψ_{2A} and ψ_{1B}:

$$\begin{aligned}
\psi_{2A}(t_{2A}) = {} & \psi(t_{2A}) + \frac{1}{4}\varepsilon k_A\Big\{\frac{1}{2}[1 + T_1 + J_A^{-1}(1 - J_A)(1 + J_A)^2K_A] \times \qquad (3.8)\\
& \psi^2(t_{2A}) - (1 - J_A)\psi^{\cdot}(t_{2A})\int_0^{t_{2A}} \psi(z)\,dz + c_A^{-1}L_A\frac{\partial}{\partial t}\psi^2(t_{2A})
\end{aligned}$$

$$\begin{aligned}
\psi_{1B}(t_{1B}) = {} & \psi(t_{1B}) + \frac{1}{8}\varepsilon k_A[1 + T_1 - (1 + J_A)^2K_A]\psi^2(t_{1B}) + \qquad (3.9)\\
& \frac{1}{4}\varepsilon k_A J_A\psi^{\cdot}(t_{1B})\int_0^{t_{1B}} \psi(z)\,dz + \frac{1}{4}\varepsilon k_A c_A^{-1}L_A\frac{\partial}{\partial t}\psi^2(t_{1B})
\end{aligned}$$

Here and in what follows we use the definition

$$K_A = \frac{k_B \beta_A}{k_A \beta_B} - 1 \qquad (3.10)$$

The substitution of (3.8) into (3.6) and of (3.9) into (3.7) yields the final formulas for calculating the first approximations $U_{A(2)1}$ and $U_{B(1)1}$ of pulses $U_{A(2)}$ and $U_{B(1)}$. The differentiation of these equations readily yields formulas for first approximations of derivatives $U_{A(2)1}$ and $U_{B(1)1}$. The formulas for computing $\dot{U}_{A(2)1}$ and $U_{B(1)1}$ are

$$\dot{U}_{A(2)1}(X, t) = [H(t_{2A}) - H(t_{2A} - t_0)]\,[\dot{V}_{A(2)0}(X, t) + \dot{V}_{A(2)1}(X, t)] + [H(t_{2A}) - H(t_{2A} - t_0)]\,[H(t_{1A}) - H(t_{1A} - t_0]\,\dot{V}_{A(12)1}(X, t) \qquad (3.11)$$

$$\dot{U}_{B(1)1}(X, t) = [H(t_{1B}) - H(t_{1B} - t_0)]\,[\dot{V}_{B(1)0}(X, t) + \dot{V}_{B(1)1}(X, t)] \qquad (3.12)$$

where

$$\dot{V}_{A(2)0}(X, t) = c_A \varepsilon J_A \psi(t_{2A}) \qquad (3.13)$$

$$\dot{V}_{A(2)1}(X, t) = c_A \varepsilon^2 k_A \Big\{ \frac{1}{8}[(1 - J_A)(1 + J_A)^2 K_A + J_A(1 + T_1)] \times \psi^2(t_{2A}) + \frac{1}{4} J_A \dot{\psi}(t_{2A}) \Big[\int_0^{t_0} \psi(z)\,dz - (1 - J_A) \int_0^{t_{2A}} \psi(z)\,dz \Big] + \frac{1}{4} c_A^{-1} [J_A^2 (L_A - X) + J_A L_A] \frac{\partial}{\partial t} \psi^2(t_{2A}) \Big\}$$

$$\dot{V}_{A(12)1}(X, t) = c_A \varepsilon^2 k_A J_A \Big\{ -\frac{1}{4} \dot{\psi}(t_{1A}) \int_0^{t_{2A}} \psi(z)\,dz + \frac{1}{4} \dot{\psi}(t_{2A}) \int_{t_0}^{t_{1A}} \psi(z)\,dz \Big\}$$

$$\dot{V}_{B(1)0}(X, t) = -c_A \varepsilon (1 - J_A) \psi(t_{1B}) \qquad (3.14)$$

$$\dot{V}_{B(1)1}(X, t) = -c_A \varepsilon^2 k_A \Big\{ \frac{1}{8}[1 + T_1 - (1 + J_A) K_A]^2 \psi^2(t_{1B}) + \frac{1}{4} J_A \dot{\psi}(t_{1B}) \int_0^{t_{1B}} \psi(z)\,dz + \frac{1}{4}[c_A^{-1} L_A + (1 + J_A)(K_A + 1) c_B^{-1} (X - L_A)] \frac{\partial}{\partial t} \psi^2(t_{1B}) \Big\}$$

Note that in formula (3.11) function $\dot{V}_{A(2)0}$ defines the zero (linear) approximation of $\dot{U}_{A(2)}$, function $\dot{V}_{A(2)1}$ defines the nonlinear component of $\dot{U}_{A(2)}$ outside the region of interaction between pulses $U_{A(1)}$ and $U_{A(2)}$, and function $\dot{V}_{A(12)1}$ determines the nonlinear component of $\dot{U}_{A(2)}$ which is nonzero only in the region of interaction between pulses $U_{A(1)}$ and $U_{A(2)}$ (see Fig. 2) and together with function $\dot{V}_{A(2)1}$ determines the nonlinear distortion of $\dot{U}_{A(2)}$ in that region.

Similarly, in formula (3.12) function $\dot{V}_{B(1)0}$ determines the zero (linear) approximation of $\dot{U}_{B(1)}$, and function $\dot{V}_{B(1)1}$ determines the nonlinear component of $\dot{U}_{B(1)}$.

We point out that the presented first approximation of the considered pulses is based on the calculation of functions $G_{A(2)1}$ and $G_{B(1)1}$ by the zero approximation (1.11) of these pulses. To determine second approximations of pulses $U_{A(2)}$ and $U_{B(1)}$ it is necessary to calculate functions $G_{A(2)2}$ and $G_{B(1)2}$ by the first approximation of these pulses, as shown in this Section.

4. Asymptotic approximation of pulses $U_{B(2)}$ and $U_{C(1)}$. First approximation formulas for pulses $U_{B(2)}$ and $U_{C(1)}$ and their derivatives can be readily derived by a procedure analogous to that described in Sect. 3. Omitting cumbersome intermediate operations, we present the final formulas for calculating the first approximation $\dot{U}_{B(2)1}$ of the quantity $\dot{U}_{B(2)}$

$$\begin{aligned}\dot{U}_{B(2)1}(X,t) = {} & [H(t_{2B}) - H(t_{2B}-t_0)]\,[\dot{V}_{B(2)0}(X,t) + \\ & \dot{V}_{B(2)1}(X,t)] + [H(t_{2B}) - H(t_{2B}-t_0)]\,[H(t_{1B}) - \\ & H(t_{1B}-t_0)]\,\dot{V}_{B(12)1}(X,t)\end{aligned} \tag{4.1}$$

where

$$\dot{V}_{B(2)0}(X,t) = c_A\varepsilon(1-J_A)J_B\psi(t_{2B}) \tag{4.2}$$

$$\begin{aligned}\dot{V}_{B(2)1}(X,t) = {} & c_A\varepsilon^2k_A(1-J_A)J_B\Big\{\frac{1}{8}[1+T_1-(1+J_A)^2K_A + \\ & J_B^{-1}(1-J_B)(1+J_B)^2(1+J_A)(K_A+1)K_B]\,\psi^2(t_{2B}) + \\ & \frac{1}{4}(1-J_A)(K_A+1)\psi^{\cdot}(t_{2B})\int_0^{t_0}\psi(z)\,dz + \\ & \frac{1}{4}[J_A-(1-J_B)(1+J_A)(K_A+1)]\,\psi^{\cdot}(t_{2B})\int_0^{t_{2B}}\psi(z)\,dz + \\ & \frac{1}{4}[c_A^{-1}L_A+(1+J_A)(K_A+1)c_B^{-1}(L_B-L_A) + \\ & J_B(1+J_A)(K_A+1)c_B^{-1}(L_B-X)]\frac{\partial}{\partial t}\psi^2(t_{2B})\Big\}\end{aligned}$$

$$\begin{aligned}\dot{V}_{B(12)1}(X,t) = {} & c_A\varepsilon^2k_A(1-J_A)(1+J_A)J_B(K_A+1)\Big\{-\frac{1}{4}\psi^{\cdot}(t_{1B})\times \\ & \int_0^{t_{2B}}\psi(z)\,dz + \frac{1}{4}\psi^{\cdot}(t_{2B})\int_{t_0}^{t_{1B}}\psi(z)\,dz\Big\}\end{aligned}$$

where, similarly to (3.10),

$$K_B = \frac{k_C\beta_B}{k_B\beta_C} - 1 \tag{4.3}$$

In formula (4.1) $\dot{V}_{B(2)0}$ defines the zero (linear) approximation of $\dot{U}_{B(2)}$, $\dot{V}_{B(2)1}$ defines the nonlinear distortion of $\dot{U}_{B(2)}$ outside the region of interaction between the incident pulse $U_{B(1)}$ and the reflected pulse $V_{B(2)}$, and $\dot{V}_{B(12)1}$ determine that part of the nonlinear distortion of $\dot{U}_{B(2)}$ which is nonzero only in the region of interaction of pulses $U_{B(1)}$ and $U_{B(2)}$.

5. Asymptotic approximation of pulses $U_{B(3)}$ and $U_{AB(1)}$. Formulas for the first asymptotic approximations for pulses $U_{B(3)}$ and $U_{AB(1)}$ (see Fig. 2) can be derived by a procedure analogous to that described in Sect. 3. For brevity we present here only the final formula for determining the first approximation $\dot{U}_{AB(1)1}$ of the quantity $\dot{U}_{AB(1)}$

$$\begin{aligned}
&\dot{U}_{AB(1)1}\,(X,\,t) = [H\,(t_{1AB}) - H\,(t_{1AB} - t_0)] \times \qquad (5.1)\\
&\quad [\dot{V}_{AB(1)0}\,(X,\,t) + \dot{V}_{AB(1)1}\,(X,\,t)]\\
&\dot{V}_{AB(1)0}\,(X,\,t) = c_A\varepsilon J_B\,(1 - J_A{}^2)\,\psi\,(t_{1AB})\\
&\dot{V}_{AB(1)1}\,(X,\,t) = c_A\varepsilon^2 k_A J_B\,(1 - J_A^2)\left\{\frac{1}{8}\,[1 + T_1 - (1 + J_A)^2\,K_A + \right.\\
&J_B^{-1}(1 - J_B)(1 + J_B)^2\,(1 + J_A)\,(K_A + 1)\,K_B + J_B\,(1 + J_A) \times\\
&(1 - J_A)^2 K_A]\psi^2(t_{1AB}) + \frac{1}{4}\,(1 + J_A)(K_A + 1)\,\psi^{\cdot}(t_{1AB})\int\limits_0^{t_0}\psi\,(z)\,dz +\\
&\frac{1}{4}\,[J_A - (1 - J_B)(1 + J_A)(K_A + 1) - J_B\,(1 + J_A)J_A\,(K_A + 1)] \times\\
&\psi^{\cdot}\,(t_{1AB})\int\limits_0^{t_{1AB}}\psi\,(z)\,dz + \frac{1}{4}\,[c_A^{-1}L_A + (1 + J_B)(1 + J_A)(K_A + 1) \times\\
&\left. c_B^{-1}\,(L_B - L_A) + J_B\,(1 - J_A^2)\,c_A^{-1}\,(L_A - X)]\,\frac{\partial}{\partial t}\,\psi^2\,(t_{1AB})\right\}
\end{aligned}$$

6. Information obtainable from the nonlinear distortion of reflected pulses entering medium A. We consider an idealized experimental situation on the following assumptions. First, the mathematical model defined in Sect. 1 is considered adequate. Second, that by a suitable selection of function $\psi\,(t)$ which defines the time dependence of interaction at the boundary $X = 0$ it is possible to decompose reflected pulses in medium A in linear and nonlinear components that vary differently in time, and to determine the amplitudes of these components.

We shall show what information about the properties of media A, B and C can be obtained on the above assumptions from the nonlinear distortion of reflected pulses which reach medium A after passing through the interfaces of media A and B, and B and C.

Let us assume that at point $X = a$ with $a = \mathrm{const}$ of medium A outside the regions of interaction between pulse $U_{A(3)}$ and pulses $U_{A(2)}$ and $U_{AB(1)}$ (see Fig. 2) functions $\dot{U}_{A(2)}\,(a,\,t) = \mathscr{E}_1\,(t)$ and $\dot{U}_{AB(1)}\,(a,\,t) = \mathscr{E}_2\,(t)$, are registered and decomposed. On the basis of (3.11), (3.13) and (5.1) we have for these functions the following first approximation asymptotic representation:

$$\mathscr{E}_j(t) = [H(t - r_j) - H\,(t - t_0 - r_j)]\left\{R_{j0}\psi\,(t - r_j) + \sum_{n=1}^{4} R_{jn}F(t - r_j)\right\} \qquad (6.1)$$

where

$$\begin{aligned}
&r_1 = c_A{}^{-1}\,(2L_A - a)\\
&R_{10} = c_B\varepsilon J_A,\quad R_{11} = {}^1/_8 c_A\varepsilon^2 k_A\,[J_A\,(1 + T_1) + (1 - J_A)\,(1 + \qquad (6.2)\\
&\quad J_A)^2 K_A],\quad R_{12} = {}^1/_4 c_A\varepsilon^2 k_A J_A M\\
&R_{13} = -\,{}^1/_4 c_A\varepsilon^2 k_A J_A\,(1 - J_A)\\
&R_{14} = {}^1/_4\varepsilon^2 k_A J_A[J_A\,(L_A - a) + L_A]\\
&r_2 = r_1 + 2c_B{}^{-1}\,(L_B - L_A)\\
&R_{20} = c_A\varepsilon\,(1 - J_A{}^2)J_B \qquad (6.3)\\
&R_{21} = {}^1/_8 c_A\varepsilon^2 k_A\,(1 - J_A{}^2)[J_B\,(1 + T_1) - J_B\,(1 + J_A)^2 K_A +\\
&\quad (1 - J_B)(1 + J_B)^2(1 + J_A)(K_A + 1)K_B + J_B{}^2\,(1 + J_A)(1 -\\
&\quad J_A)^2 K_A]
\end{aligned}$$

$$R_{22} = {}^1\!/_4 c_A \varepsilon^2 k_A J_B (1 - J_A^2)(1 + J_A)(K_A + 1)M$$
$$R_{23} = {}^1\!/_4 c_A \varepsilon^2 k_A J_B (1 - J_A^2)[J_A - (1 - J_B)(1 + J_A)(K_A + 1) - J_B (1 + J_A) J_A (K_A + 1)]$$
$$R_{24} = {}^1\!/_4 c_A \varepsilon^2 k_A J_B (1 - J_A^2)[c_A^{-1} L_A + J_B (1 - J_A^2) c_A^{-1} (L_A - a) + (1 + J_B)(1 + J_A)(K_A + 1) c_B^{-1} (L_B - L_A)]$$
$$F_1(t) = \psi^2(t), \quad F_2(t) = \psi^{\cdot}(t) \tag{6.4}$$
$$F_3(t) = \psi^{\cdot}(t) \int_0^t \psi(z)\,dz, \quad F_4(t) = \frac{\partial}{\partial t} \psi^2(t)$$
$$M = \int_0^{t_0} \psi(z)\,dz \tag{6.5}$$

Note that functions (6.4) and the integral (6.5) are determined by specifying function $\psi(t)$, i.e. by time dependence of the interaction.

In conformity with the assumptions formulated at the beginning of this Section we consider r_i and R_{ij} $(i = 1, 2;\ j = 0, 1, 2, 3, 4)$ to be constants obtained by processing experimental data.

Formulas (6.2) show that the six constants r_1 and R_{1j} $(j = 0, 1, 2, 3, 4)$ which are coefficients of the first approximation of function $\mathscr{E}_1(t)$ are expressed in terms of five parameters c_A, L_A, J_A, K_A and k_A of the layered medium. The resulting from this "overdetermination" of the inverse problem of calculating c_A, L_A, J_A, K_A and k_A by r_1 and R_{1j} $(j = 0, 1, 2, 3, 4, 5)$ vanishes only in the particular case when function $\psi(t)$ is specified so that the integral (6.5) vanishes and, consequently, $R_{12} = 0$. However, owing to the smallness of constant R_{12}, it is not expedient to use it for determining the layered medium parameters.

It follows from formulas (6.3) that the six constants r_2 and R_{2j} $(j = 0, 1, 2, 3, 4, 5)$ which are the coefficients of the first approximation of function $\mathscr{E}_2(t)$ are expressed in terms of the following nine parameters of the layered medium: c_A, L_A, J_A, K_A, k_A, c_B, L_B, J_B and K_B.

Let the amplitude ε and the time dependence $\psi(t)$ of interaction be known. Then, with allowance for formulas (1.3), (1.14), (3.10) and (4.3), we come to the conclusion that the time of arrival (r_1, r_2) at point $X = a$ of pulses $\dot{U}_{A(2)}$ and $\dot{U}_{AB(1)}$ and amplitude (R_{10}, R_{20}) and of their linear components makes it possible to determine the numerical values of the four quantities

$$\beta_B \rho_B / \beta_A \rho_A, \quad (L_A - a)(\rho_A / \beta_A)^{1/2}$$
$$\beta_C \rho_C / \beta_B \rho_B, \quad (L_B - L_A)(\rho_B / \beta_B)^{1/2}$$

and, if the amplitudes R_{ij} $(i = 1, 2;\ j = 1, 2, 3, 4)$ of the first approximations of the nonlinear components of these pulses are used, it is possible to determine the following nine parameters of the layered medium:

$$\beta_B \rho_B / \beta_A \rho_A, \quad \beta_A / \rho_A, \quad L_A, \quad k_A, \quad k_B \beta_A / k_A \beta_B$$
$$\beta_C \rho_C / \beta_B \rho_B, \quad \beta_B / \rho_B, \quad L_B, \quad k_C \beta_B / k_B \beta_C$$

It should be particularly stressed that the nonlinear theory makes it possible to calculate separately the thickness of the propagation velocity of waves.

Some of the results presented here were earlier given by the author in [3, 9, 10]. The problem of nonlinear distortion of pulses in a layered medium were investigated in [11, 12] from a different point of view.

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